# Forces on a dielectric slab inserted into a capacitor

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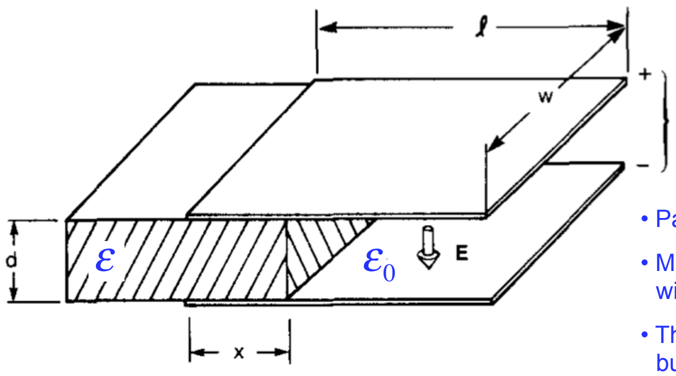
# The problem setting

## Dielectric slab inserted into a capacitor

### The proper setup




$\Delta\varphi = \text{const.} := U$

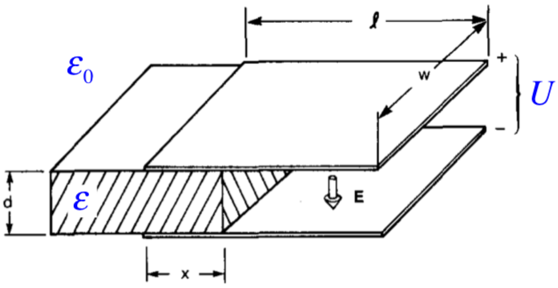
- Parallel-plate configuration
- Movable dielectric slab with permittivity $\varepsilon$.
- The plates are at different but constant electric potentials.

⇒ Calculate the force excerted on the slab

# The standard textbook calculation I

## Dielectric slab inserted into a capacitor

### The principle of virtual work




(A) Energy within the system:

The field is assumed to be homogeneous and the fringe fields are therefore neglected.

$$W_{el} = \tfrac{1}{2}\iiint_V \vec{E}\cdot\vec{D}\cdot dV =$$
$$= \tfrac{1}{2}\iiint_V \varepsilon\left|\vec{E}\right|^2\cdot dV =$$
$$= \tfrac{1}{2}\varepsilon\left(\tfrac{U}{d}\right)^2 xwd +$$
$$+\tfrac{1}{2}\varepsilon_0\left(\tfrac{U}{d}\right)^2(\ell - x)wd$$

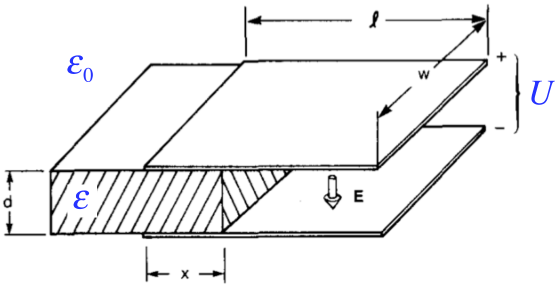
$$\underline{\underline{W_{el} = \tfrac{wd}{2}\left(\tfrac{U}{d}\right)^2\left[\varepsilon x + \varepsilon_0(\ell - x)\right]}}$$



# The standard textbook calculation II

## Dielectric slab inserted into a capacitor

### The principle of virtual work




(B) Force excerted on the slab:

We assumed constant potentials, namely $\Delta\varphi = U$ = const., hence:

$$\boxed{\vec{F} = +\,\mathrm{grad}\,W_e\Big|_{\substack{\varphi_i=\mathrm{const.}\\ U_{ij}=\mathrm{const.}}}}$$

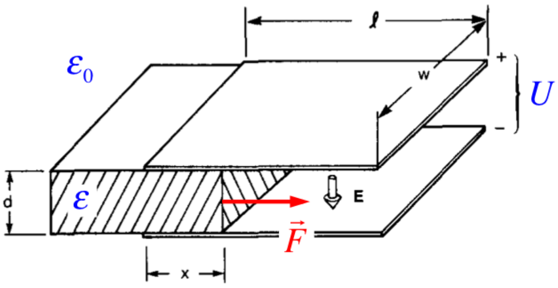
Given the previous expression for $W_e$ we get the well known textbook result:

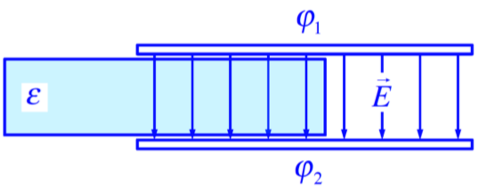
$$\vec{F} = \tfrac{d}{dx}\left\{\tfrac{wd}{2}\left(\tfrac{U}{d}\right)^2\left[\varepsilon x + \varepsilon_0(\ell - x)\right]\right\}\cdot\vec{e}_x = \underline{\underline{\tfrac{(\varepsilon-\varepsilon_0)w}{2d}U^2\cdot\vec{e}_x}}$$

# The standard textbook calculation III

## Dielectric slab inserted into a capacitor

### The principle of virtual work




(C) Discussion:

The dielectric slab experiences a constant drag into the capacitor.

$$\boxed{\vec{F} = \frac{(\varepsilon - \varepsilon_0) w}{2d} U^2 \cdot \vec{e}_x}$$

Strange fact:

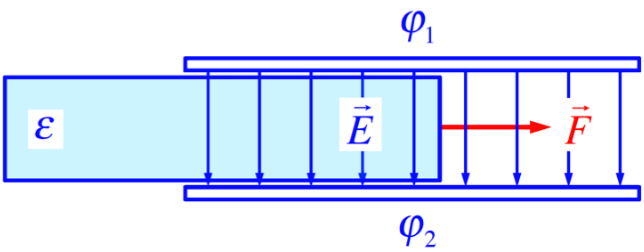
How can the emergence of a horizontal force component be physically explained when the involved field is normally impinging upon the dominant surfaces?



# The standard textbook calculation IV

## Dielectric slab inserted into a capacitor

### The open question

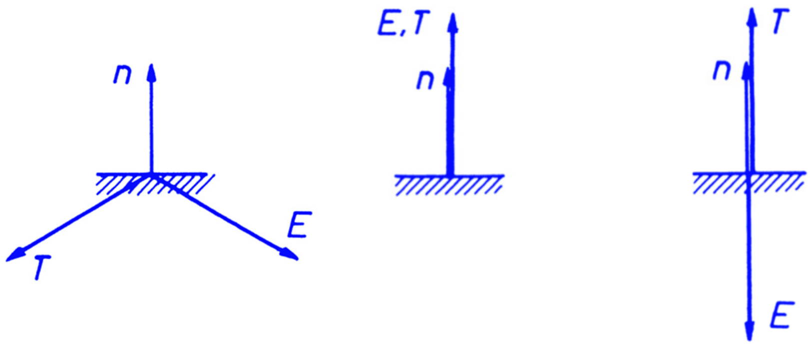



How can the force be physically explained ?

- What are your suggestions ?
- Just try to provide a reasoning !

# The Maxwell stress tensor

## Dielectric slab inserted into a capacitor




$$\boxed{\vec{F} = \oiint_{\partial V} \vec{\vec{T}}_e \cdot \vec{n} \cdot dA}$$

$$\vec{\vec{T}}_e = \varepsilon \cdot \left(\vec{E} \otimes \vec{E}\right) - \frac{\varepsilon}{2} \cdot \left(\vec{E} \cdot \vec{E}\right) \cdot \delta$$

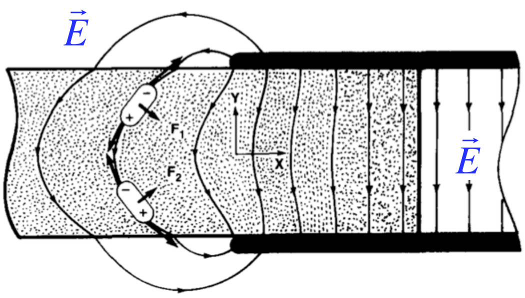
- Forces with components parallel to the slab surface are only emerging for oblique electric fields.
- The fringing field may become relevant for the explanation of the forces.



# The induced dipole argument

## Dielectric slab inserted into a capacitor

**The proper setup**

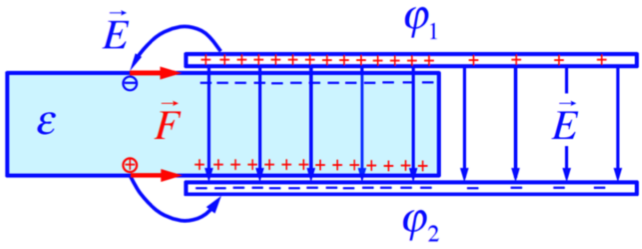



- The electric fringe field induces dipoles in the dielectric slab.
- The induced dipoles are exposed to a non-uniform electric field.
- This leads to a force according to the relation:

$$\boxed{\vec{F} = \left(\vec{p} \cdot \mathrm{grad}\right) \vec{E} = \frac{\partial \vec{E}}{\partial \vec{p}}}$$

# The induced charge argument

## Dielectric slab inserted into a capacitor




- The <u>fringe field</u> induces surface charges adjacent to the electrodes.
- The induced surface charges are <u>attracted</u> by the free charges of the electrodes.



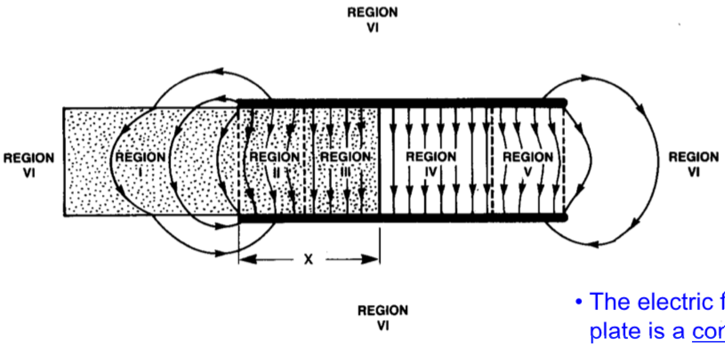
# Why is the standard textbook calculation working at all?

## Dielectric slab inserted into a capacitor

**<u>Paradox</u>**: How can the <u>standard textbook calculation</u> yield a result for the force when it is explicitly based on assuming a uniform field that exists only in the volume between the capacitor plates and <u>completely ignoring</u> the existence of a <u>fringe field</u> ?

# Why is the standard textbook calculation working at all?

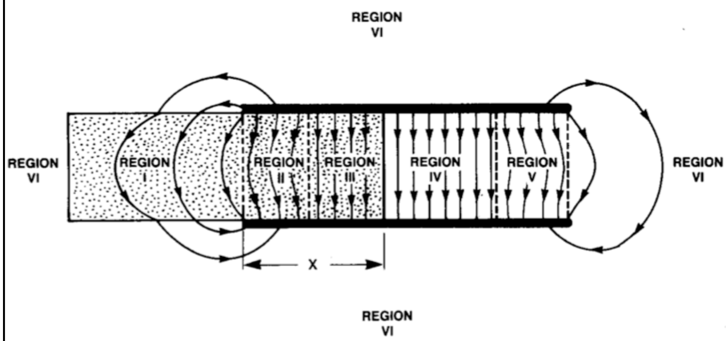
## Dielectric slab inserted into a capacitor




- The electric field between the plate is a complicated function of $x$ (cf. region II and V).
- The expression on pp. 3 does not give the total energy of the parallel plate capacitor.



# Why is the standard textbook calculation working at all?




- Special case: the right-hand edge of the slab extends into the uniform region; and the left-side edge extends beyond the finge field.
- Assume an infinitesimal displacement $dx$ to the right.
- The change of total energy is then:

$$dW_{el}^{(tot)} = \overbrace{dW_{el}^{(I)}}^{\approx 0} + \overbrace{dW_{el}^{(II)}}^{\approx 0} + \overbrace{dW_{el}^{(III)}}^{\uparrow} + \overbrace{dW_{el}^{(IV)}}^{\downarrow} + \overbrace{dW_{el}^{(V)}}^{\approx 0} + \overbrace{dW_{el}^{(VI)}}^{\approx 0} =$$

$$dW_{el}^{(tot)} = \tfrac{\varepsilon}{2}\left(\tfrac{U}{d}\right)^2 (wd)\,dx - \tfrac{\varepsilon_0}{2}\left(\tfrac{U}{d}\right)^2 (wd)\,dx = \underline{\underline{\tfrac{(\varepsilon-\varepsilon_0)w}{2d} U^2 dx}}$$

# Why is the standard textbook calculation working at all?

## Dielectric slab inserted into a capacitor

$$dW_{el}^{(tot)} = \frac{(\varepsilon-\varepsilon_0)w}{2d} U^2 dx$$

$$\left|\vec{F}\right| = \frac{dW_{el}^{(tot)}}{dx} = \frac{(\varepsilon-\varepsilon_0)w}{2d} U^2$$

$$\boxed{\vec{F} = \frac{dW_{el}^{(tot)}}{dx} \cdot \vec{e}_x = +\operatorname{grad} W_{el}^{(tot)} = \frac{(\varepsilon-\varepsilon_0)w}{2d} U^2 \cdot \vec{e}_x}$$

- Conclusion: The special case – and only the special case – yields the same result for the force as known from the principle of virtual work.
- Or: All standard textbook calculations of the force are implicitly refering to the special case.



# Facit

## Dielectric slab inserted into a capacitor

The fringing field of the papallel-plate capacitor plays a constitutive role in the explanation of the emergence of forces that are excerted on a dielectric slab between the two plates!

Further reading:

S. Margulies, «Force on a dielectric slab inserted into a parallel-plate capacitor», *Am. J. Phys.*, vol. 52, no. 6, pp.515-518, June 1984.