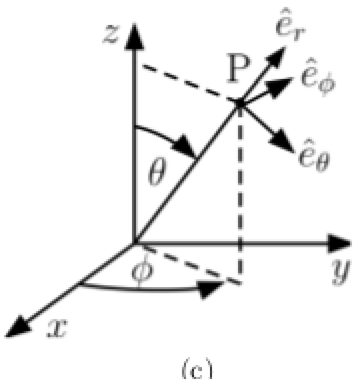
Coordenadas esféricas

Esféricas:
$0 < r < \infty$
$0 < \theta < \pi$
$0 < \phi < 2\pi$

- Simetría azimutal: $\varphi(r,\theta,\phi) = \varphi(r,\theta)$

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right) = 0.$$



$$x = \cos\theta,\ \ \varphi(r,x) = R(r)M(x)$$

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \kappa R = 0$$

$$\kappa = \nu(\nu+1)$$

$$R_\nu(r) = A_\nu r^\nu + B_\nu r^{-(\nu+1)}$$

$$x = \cos\theta \Rightarrow d/d\theta = -\sin\theta\cdot d/dx$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dM}{dx}\right] + \kappa M = 0.$$

$$(x^2-1)\frac{d^2M}{dx^2} + 2x\frac{dM}{dx} - \nu(\nu+1)M = 0$$

Ecuación de Legendre

Tiene solución para valores arbitrarios de ν: funciones asociadas de Legendre de primer $P_\nu(x)$ y segunda $Q_\nu(x)$ especie. Pero son divergentes para x=+/-1, en el eje z.

Se arregla para ν natural: $\ell = 0, 1, 2, \ldots$ $\quad P_\ell(\cos\theta)$

Los polinomios de Legendre son las soluciones finitas en el eje z

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right]P_\ell(\cos\theta)$$

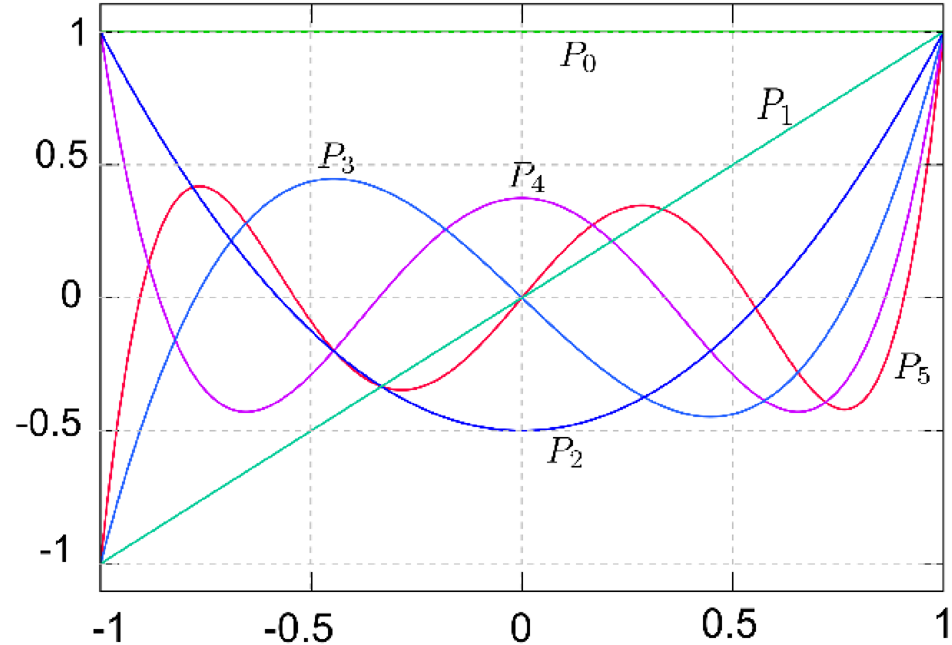
Coordenadas esféricas

- Polinomios de Legendre

Fórmula de Rodrigues $P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell$

$$\mathrm{P}_\ell(x) = \sum_{r=0}^{[\frac{1}{2}\ell]} \frac{(-1)^r}{2^\ell r!} \frac{(2\ell-2r)!x^{\ell-2r}}{(\ell-r)!(\ell-2r)!}$$

par $[\frac{1}{2}\ell] = \frac{1}{2}\ell$ impar $[\frac{1}{2}\ell] = \frac{1}{2}(\ell-1)$



$P_\ell(-x) = (-1)^\ell P_\ell(x)$ (paridad)

$|P_\ell(x)| \leq 1$ $\quad -1 < x < 1$

Relaciones de ortogonalidad y completitud

$$\langle P_\ell, P_n \rangle = \int_{-1}^{1} P_\ell(x)P_n(x)dx = \int_0^\pi P_\ell(\cos\theta)P_n(\cos\theta)\,\mathrm{sen}\,\theta d\theta = \frac{2}{2\ell+1}\delta_{\ell n}$$

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(y)P_\ell(x) = \delta(y-x)$$

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \tfrac{1}{2}(3x^2-1)$$

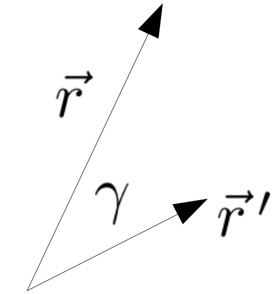
Serie de Fourier-Legendre:

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x) \qquad a_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} f(x)P_\ell(x)dx$$

$$= \frac{2\ell+1}{2^{\ell+1}l!}\int_{-1}^{1}(1-x^2)^\ell \frac{d^\ell f}{dx^\ell}dx \quad \left(\text{si } f \in C^\ell(-1,1)\right)$$

Ver todas las propiedades en el “manual” de fórmulas en el EVA

Desarrollo del 1/R en Polinomios de Legendre -(apuntes_c8_3)



$$\frac{1}{|\vec r - \vec r\,'|} = \frac{1}{\sqrt{r^2 + r'^2 - 2\vec r\cdot\vec r\,'}} = \frac{1}{r_>\sqrt{1+\left(\frac{r_<}{r_>}\right)^2 - \frac{2\vec r_>\cdot\vec r_<}{r_>^2}}}$$

$t \leqslant 1$

$$\frac{1}{\sqrt{1-2t\cos\gamma + t^2}} = \sum_{n=0}^{\infty} t^n \mathrm{P}_n(\cos\gamma)$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty}\binom{\alpha}{n}x^n \qquad \binom{\alpha}{n} = \prod_{k=1}^{n}\frac{\alpha-k+1}{k}$$

$$\mathrm{F}(x,t) = (1-2xt+t^2)^{-1/2}$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$(1-\epsilon)^{-1/2} = \sum_{n=0}^{\infty}\frac{1}{2^n}\frac{(2n-1)!!}{n!}\epsilon^n = \sum_{n=0}^{\infty}\frac{(2n)!\epsilon^n}{2^{2n}(n!)^2}$$

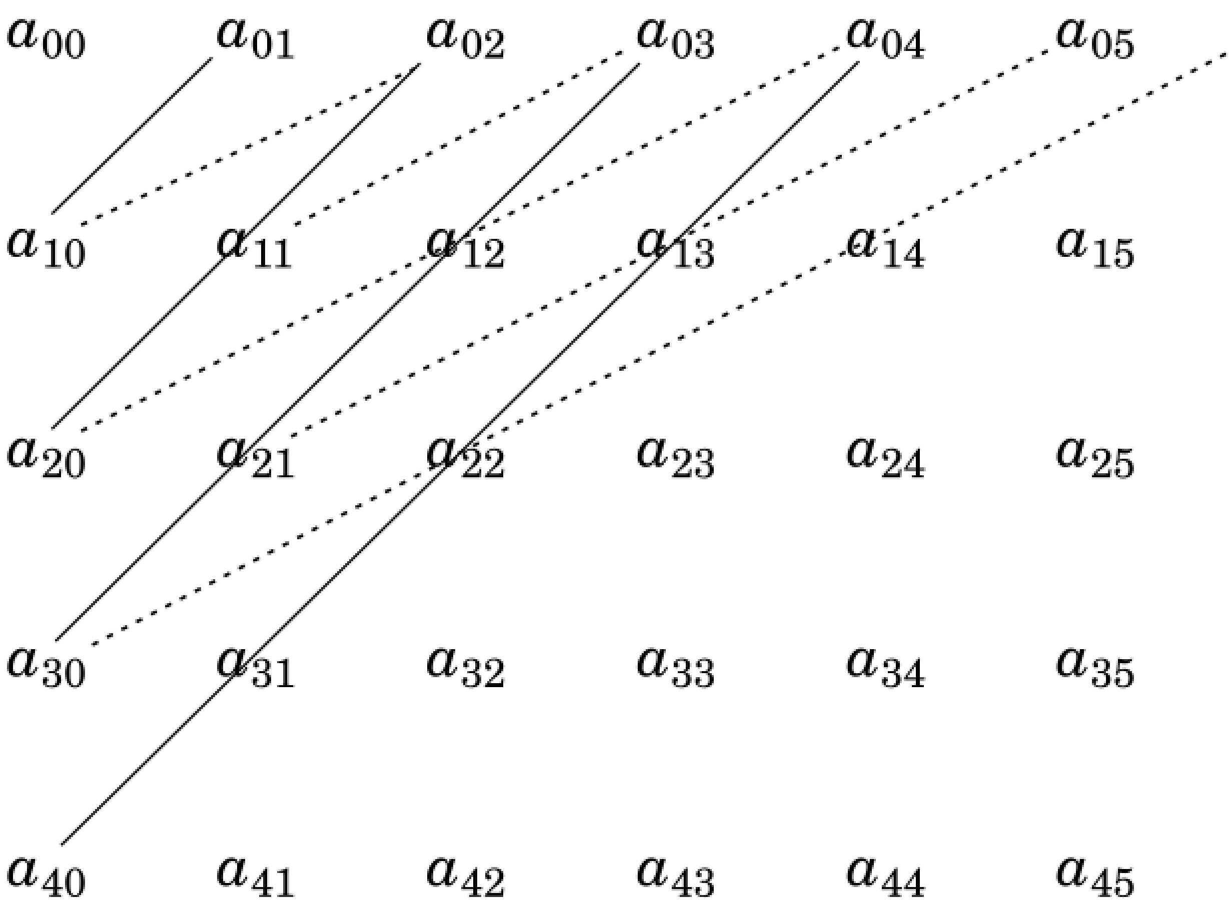
$$[1-t(2x-t)]^{-1/2} = \sum_{n=0}^{\infty}\frac{(2n)!\,t^n(2x-t)^n}{2^{2n}(n!)^2}$$

$$(2x-t)^n = \sum_{k=0}^{n}\frac{(-1)^k n!\,(2x)^{n-k}t^k}{k!\,(n-k)!}$$

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(2n)!\,(2x)^{n-k}t^{n+k}}{2^{2n}n!\,k!\,(n-k)!}$$

$$a_{k,n-k} = \frac{(-1)^k(2n)!\,(2x)^{n-k}t^{n+k}}{2^{2n}k!\,n!\,(n-k)!}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{k,n-k} = a_{0,0} + (a_{0,1} + a_{1,0}) + (a_{0,2} + a_{1,1} + a_{2,0}) + \cdots$$



$$a_{k,n-k} = \frac{(-1)^k (2n)!\,(2x)^{n-k} t^{n+k}}{2^{2n} k!\, n!\, (n-k)!}$$

n → n-k

n-k → n-2k

$$\frac{(-1)^k (2n-2k)!\, x^{n-2k} t^n}{2^n k!\, (n-k)!\, (n-2k)!}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]} a_{k,n-2k} = a_{0,0} + a_{0,1} + (a_{0,2} + a_{1,0}) + (a_{0,3} + a_{1,1})$$
$$+ (a_{0,4} + a_{1,2} + a_{2,0}) + (a_{0,5} + a_{1,3} + a_{2,1}) \cdots$$

Desarrollo del 1/R en Polinomios de Legendre

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{[\frac{1}{2}n]} \frac{(-1)^k (2n-2k)!\, x^{n-2k}}{2^n k!\,(n-k)!\,(n-2k)!}\right] t^n \quad = \sum_{n=0}^{\infty} \mathrm{P}_n(x) t^n$$

$$\mathrm{P}_\ell(x) = \sum_{r=0}^{[\frac{1}{2}\ell]} \frac{(-1)^r}{2^\ell r!} \frac{(2\ell-2r)!x^{\ell-2r}}{(\ell-r)!(\ell-2r)!}$$

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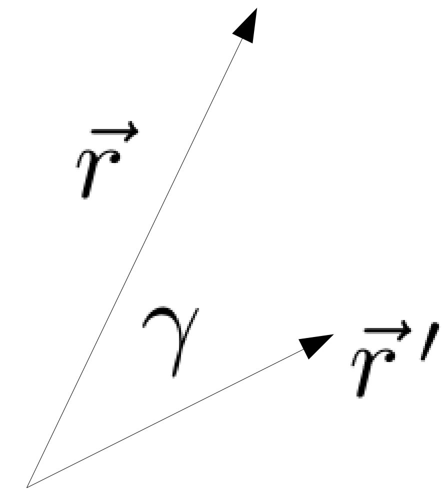

t = $\left(\frac{r_<}{r_>}\right)$ X = $\cos\gamma$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{\sqrt{r^2+r'^2-2\vec{r}\cdot\vec{r}\,'}} = \frac{1}{r_>\sqrt{1+\left(\frac{r_<}{r_>}\right)^2 - \frac{2\vec{r}_>\cdot\vec{r}_<}{r_>^2}}}$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{1}{r_>}\sum_{n=0}^{\infty}\left(\frac{r_<}{r_>}\right)^n \mathrm{P}_n(\cos\gamma)$$